Lecture 8 : Integration By Parts

Recall the product rule from Calculus 1:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

We can reverse this rule to get a rule of integration:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

or

$$\int udv = uv - \int vdu.$$

The definite integral is given by:

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x)dx$$

Example Find $\int x\cos(2x)dx$ $\quad \int_0^2 xe^x dx$

Trick : Letting $dv = dx$ $\int \ln x dx$ $\int_{-2}^{2} \ln(x+3)dx$

Using Integration by parts twice $\int x^2 \cos x \mathrm{dx}$ $\int (\ln x)^2 dx$

Recurring Integrals $\int e^{2x}\cos(5x)dx$

Powers of Trigonometric functions Use integration by parts to show that

$$\int \sin^5 x dx = \frac{-1}{5}[\sin^4 x \cos x - 4\int \sin^3 x dx]$$

This is an example of the reduction formula shown on the next page.
(Note we can easily evaluate the integral $\int \sin^3 x dx$ using substitution; $\int \sin^3 x dx = \int \sin^2 x \sin x dx = \int (1-\cos^2 x)\sin x dx$.)

In fact, we can deduce the following general **Reduction Formulae for** $\sin$ **and** $\cos$**:**

For $n \geq 2$ We can find the integral $\int \cos^n x dx$ by reducing the problem to finding $\int \cos^{n-2} x dx$ using the following reduction formula:

$$\boxed{\int \cos^n x = \frac{1}{n}[\cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx]}$$

We prove this using integration by parts:

$$u = \cos^{n-1} x \qquad dv = \cos x dx$$

$$du = -(n-1)\cos^{n-2} x \sin x \qquad v = \sin x$$

$$\int \cos^n x = \int \cos^{n-1} \cos x dx = \cos^{n-1} x \sin x - (n-1) \int \sin^2 x \cos^{n-2} x dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx$$

$$n \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$$

multiply by $\frac{1}{n}$ to get the formula.

We have a similar reduction formula for integrals of powers of sin: (you should prove this using integration by parts)

$$\boxed{\int \sin^n x dx = \frac{-1}{n}[\sin^{n-1} x \cos x - (n-1) \int \sin^{n-2} x dx]}$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sin^0 x dx = \int 1 dx = x + C$$

Extra Examples for the Enthusiast

$$\int \cos(\ln x)dx$$

$$\int \cos^3 x dx$$

$$\int \sin^4 x dx$$

Recurring Integrals $\int \cos(\ln x)dx$
Let $u = \cos(\ln x), \quad dv = dx$
Then $du = \frac{-\sin(\ln x)}{x}dx$ and $v = x$
$\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx$

We work on $\int \sin(\ln x)dx$:
Let Let $u = \sin(\ln x), \quad dv = dx$
Then $du = \frac{\cos(\ln x)}{x}dx$ and $v = x$
$\int \sin(\ln x)dx = x\sin(\ln x) - \int \frac{\cos(\ln x)}{x}xdx = x\sin(\ln x) - \int \cos(\ln x)dx$

Now substituting this for $\int \sin(\ln x)dx$ in the equation above we get:
$\int \cos(\ln x)dx = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)dx$
Taking all multiples of $\int \cos(\ln x)dx$ to the Left Hand side we get
$2\int \cos(\ln x)dx = x\cos(\ln x) + x\sin(\ln x)$
or
$\int \cos(\ln x)dx = \frac{x\cos(\ln x)+x\sin(\ln x)}{2} + C$

Using Integration by parts for $\int \cos^3 xdx$ **(reduction formula):**

$\int \cos^3 xdx = \int \cos^2 x\cos x\ dx$
Let Let $u = \cos^2 x, \quad dv = \cos x\ dx$
Then $du = -2\cos x\sin xdx$ and $\sin x$
We get

$$\int \cos^3 xdx = \cos^2 x\sin x + 2\int \sin^2 x\cos x\ dx = \cos^2 x\sin x + 2\int \cos x(1-\cos^2 x)dx$$

$$= \cos^2 x\sin x + 2\int \cos xdx - 2\int \cos^3 xdx$$

Therefore

$$\int \cos^3 xdx + 2\int \cos^3 xdx = \cos^2 x\sin x - 2\sin x$$

or

$$\int \cos^3 xdx = \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x.$$

Using integration by parts as to figure out $\int \sin^4 xdx$ **(Reduction formula)**

$$\int \sin^4 xdx = \int \sin^3 x\sin xdx$$

Let
$u = \sin^3 x$ and $dv = \sin xdx$
$du = 3\sin^2 x\cos xdx$ and $v = -\cos x$
We have

$$\int \sin^4 xdx = \sin^3 x\cos x - \int (-\cos x)3\sin^2 x\cos xdx = \sin^3 x\cos x + \int (\cos^2 x)3\sin^2 xdx$$

$$= \sin^3 x \cos x + \int (1 - \sin^2 x) 3 \sin^2 x dx = \sin^3 x \cos x + 3 \int \sin^2 x - \sin^4 x dx$$

This gives

$$\int \sin^4 x dx = \sin^3 x \cos x + 3 \int \sin^2 x dx - 3 \int \sin^4 x dx$$

which gives

$$\int \sin^4 x dx + 3 \int \sin^4 x dx = \sin^3 x \cos x + 3 \int \sin^2 x dx$$

or

$$\int \sin^4 x dx = \frac{1}{4} \big(\sin^3 x \cos x + 3 \int \sin^2 x dx \big)$$

Thus we have reduced the problem to figuring out the integral $\int \sin^2 x dx$. You can use the half angle formula for this or try integration by parts again to reduce $\int \sin^2 x dx$ to $\int \sin^0 x dx$.
